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**A RENORMALISATION GROUP STUDY
OF
THREE DIMENSIONAL TURBULENCE**

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Abstract:

We study the three dimensional Navier-Stokes equation with a random Gaussian force acting on large wavelengths. Our work has been inspired by Polyakov's analysis of steady states of two dimensional turbulence. We investigate the time evolution of the probability law of the velocity potential. Assuming that this probability law is initially defined by a statistical field theory in the basin of attraction of a renormalisation fixed point, we show that its time evolution is obtained by averaging over small scale features of the velocity potential. The probability law of the velocity potential converges to the fixed point in the long time regime. At the fixed point, the scaling dimension of the velocity potential is determined to be $-\frac{4}{3}$. We give conditions for the existence of such a fixed point of the renormalisation group describing the long time behaviour of the velocity potential. At this fixed point, the energy spectrum of three dimensional turbulence coincides with a Kolmogorov spectrum.

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Turbulence is one of the crucial puzzles of theoretical Physics. Despite its practical relevance, there is still no real thorough understanding of the principles of turbulence. A few years ago Polyakov suggested using conformal field theories to study turbulence in two dimensions^[1]. Polyakov's idea is that the scale invariant regime of turbulence can be described by a conformal field theory. In three dimensions, perturbative renormalisation transformations in momentum space have been used to derive an infrared fixed point describing the long distance physics of the Navier-Stokes equation^[2,3,4]. Similarly, probabilistic solutions of the Navier-Stokes equation with no forcing term have been provided by self-similar probability laws for the velocity field^[5,6]. We shall use properties of the Navier-Stokes equation under scaling transformations in order to study the time evolution of solutions. In particular, our analysis will combine non-perturbative statements about the renormalisation group and results about 3d conformal field theories applied to the Navier-Stokes equation.

The motion of a fluid is described by the macroscopic velocity field. We shall suppose that the macroscopic velocity field v_a is regularised at small distances by a cut-off a . This cut-off is related to the viscosity by

$$\nu = \frac{a^2}{\tau} \quad (1)$$

where τ is a time characterising the energy decay when a stirring force is absent. For incompressible fluids, the velocity field is divergenceless:

$$\nabla \cdot v_a = 0. \quad (2)$$

This implies that the velocity field can be written

$$v_a = \text{curl } \psi_a \quad (3)$$

where ψ_a is the velocity potential. We will be interested in the vorticity

$$\omega_a = \text{curl } v_a. \quad (4)$$

The vorticity satisfies the vorticity Navier-Stokes equation

$$\frac{\partial \omega_a}{\partial t} + (v_a \cdot \nabla) \omega_a = (\omega_a \cdot \nabla) v_a + \nu \Delta \omega_a + M_a. \quad (5)$$

where $M = \text{curl} F$ is the vorticity forcing term and F is the velocity forcing term. In order to discard boundary effects, we shall only be interested in the case where the fluid fills up all space. Furthermore, we shall require the vorticity to decrease at infinity. The initial conditions and the external stirring are chosen to be regularised random fields where the ultra violet cut-off is given by the scale a . In particular, the velocity forcing term will be supposed to be Gaussian with a white noise temporal dependence and an action over large distances. Taking into account the incompressibility condition, its 2-point correlation function is

$$\langle F_{a,\alpha}(k, \omega) F_{a,\beta}(k', \omega') \rangle = W_0 S_L(k) \left(\delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2} \right) \delta(k + k') \delta(\omega + \omega') \quad (6)$$

in Fourier space. The constant W_0 determines the amplitude of the forcing while its k -dependence is defined by $S_L(k)$. We suppose that S_L is a positive smoothing function of width L^{-1} , i.e. $S_L(k) = L^3 S(Lk)$ where S_L has width L^{-1} and is centered around zero, it is also normalised by $\int d^3k S(k) = 1$. The width is supposed to be arbitrarily small. Notice that the spectrum^a of the stirring force F is $k^2 S_L(k)$ which is peaked around $k \sim L^{-1}$. As a is the only intrinsic scale, the scale L is related to a by

$$L = \kappa a \quad (7)$$

where κ is a large number depending on the forcing term. For a given initial configuration and a given realisation of the stirring force, solutions of (5) will solve the initial value Cauchy problem. Starting from an initial probability law $d\mathcal{P}_0(\psi)$, one can solve (5) for a given realisation of the stirring force. This defines the conditional probability law $d\mathcal{P}(\psi_a(x, t)|F_a(x, t))$ expressing the probability law of $\psi_a(x, t)$ for a given realisation of the stirring force F_a . This probability law is such that the Navier-Stokes equation is valid in all correlation functions

$$\begin{aligned} \frac{\partial}{\partial t} \langle \omega_a(x_1, t) \dots \omega_a(x_n, t) \rangle = \\ \sum_{i=1}^n \langle \omega_a(x_1, t) \dots (-(v_a \cdot \nabla) \omega_a + (\omega_a \cdot \nabla) v_a + \nu \Delta \omega_a + M_a) \dots \omega_a(x_n, t) \rangle \end{aligned} \quad (8)$$

for a given realisation of the stirring force F_a . These equations are the Hopf equations. The conditional probability law $d\mathcal{P}(\psi_a(x, t)|F_a(x, t))$ is random when seen as a functional of the stirring force. After averaging over the forcing term realisations, one gets the probability law of the velocity potential at time t

$$d\mathcal{P}(\psi_a(x, t)) = \int d\mathcal{P}(\psi_a(x, t)|F_a(x, t)) \otimes d\mu(F_a(x, t)) \quad (9)$$

where $d\mu(F_a(x, t))$ is the Gaussian probability law of the stirring force at time t . The main object of the present work is to characterise the long time behaviour of $d\mathcal{P}(\psi_a)$.

We shall suppose that the conditional probability at time t is defined by a statistical field theory

$$d\mathcal{P}(\psi_a(x, t)|F_a(x, t)) = \frac{1}{Z} d\psi \exp -S_{t,\tau}(\psi_a, F_a) \quad (10)$$

where the partition function Z is a normalisation factor, $d\psi$ stands for the functional integral symbol and $S_{t,\tau}(\psi_a, F_a)$ is an effective action at time t depending on the regularised fields. By dimensional analysis, the action is a homogeneous function of $\frac{t}{\tau}$. After integrating over the stirring force, one obtains the effective action describing the probability law of the velocity potential at time t

$$\begin{aligned} d\mathcal{P}(\psi_a) &= \frac{1}{Z} d\psi \exp -S_{t,\tau}(\psi_a) \\ \exp -S_{t,\tau}(\psi_a) &= \int \exp -S_{t,\tau}(\psi_a, F_a) d\mu(F_a(x, t)) \end{aligned} \quad (11)$$

^a Practically, $S(k)$ can be taken to be $\pi^{-\frac{3}{2}} \exp -k^2$, in that case $k^2 S_L(k)$ has a maximum at $k = L^{-1}$.

We shall suppose that the velocity potential is a field with scaling dimension d_ψ . This dimension is for instance well-defined if the effective field theory (11) is the basin of attraction of a fixed point of the renormalisation group. In that case the scaling dimension of ψ becomes its conformal dimension. In this communication, we shall show that the time evolution of solution of (9) starting from the initial conditions (11) is determined by renormalisation transformations. The evolution of solutions of (9) is given by

$$\begin{aligned}\tilde{\psi}_a(\lambda x, \lambda^T t) &= \lambda^{-d_\psi} \psi_{\lambda^{-1}a}(x, t) \\ \exp -S_{\lambda^T t, \tau}(\tilde{\psi}_a(\lambda x, \lambda^T t)) &= \int_{[\frac{a}{\lambda}, a]} d\psi \exp -S_{t, \lambda^{-T} \tau}(\psi_{\lambda^{-1}a}(x, t)).\end{aligned}\tag{12}$$

This is the precise statement that the time evolution of solutions follows renormalisation trajectories. The first equation implies that the velocity potential lies on the renormalisation trajectory of a field of dimension d_ψ , i.e. the solution $\tilde{\psi}_a(\lambda x, \lambda^T t)$ of the Navier-Stokes equation at time $\lambda^T t$ is simply the renormalised field $\psi_a(\lambda x, t)$ of $\psi_{\lambda^{-1}a}(x, t)$. The second equation entails that the effective action at time $\lambda^T t$ is obtained after integrating over the fluctuations in the range $[\lambda^{-1}a, a]$. This renormalisation transformation is performed in momentum space integrating over modes in $[a^{-1}, \lambda a^{-1}]$. The time rescaling exponent T and the scaling dimension of ψ are uniquely determined

$$\begin{aligned}T &= \frac{2}{3} \\ d_\psi &= -\frac{4}{3}\end{aligned}\tag{13}$$

Notice that the scaling dimension of ω is $d_\omega = \frac{2}{3}$ guaranteeing that the vorticity decreases at infinity.

Let us now prove these results. They depend crucially on properties of the stirring force F_a . Notice that the spectrum of the stirring force $S_L(k)$ has length dimension -3 . Moreover, one obtains

$$S_L\left(\frac{k}{\lambda}\right) = \lambda^3 S_{\lambda^{-1}L}(k)\tag{14}$$

Using (6), one can deduce the following scaling relation

$$\langle F_a\left(\frac{k}{\lambda}, \frac{\omega}{\lambda^T}\right), F_a\left(\frac{k'}{\lambda}, \frac{\omega'}{\lambda^T}\right) \rangle = \lambda^{6+T} \langle F_{\lambda^{-1}a}(k, \omega), F_{\lambda^{-1}a}(k', \omega') \rangle\tag{15}$$

As the stirring force is Gaussian, this relation between 2-point functions yields an equality

$$F_a(\lambda x, \lambda^T t) = \lambda^{\frac{-T}{2}} F_{\lambda^{-1}a}(x, t),\tag{16}$$

i.e. these two random variables have the same probability law. In other words, one has the equality

$$\int f(\lambda^{\frac{T}{2}} F_a(\lambda x, \lambda^T t)) d\mu(F_a(\lambda x, \lambda^T t)) = \int f(F_{\lambda^{-1}a}(x, t)) d\mu(F_{\lambda^{-1}a}(x, t))\tag{17}$$

for any functional f . This fact will allow us to relate the probability law of the stream function at two different times when taking the average of (19). Let us rewrite the Navier-Stokes equation after rescaling $a \rightarrow \lambda^{-1}a$ and $\tau \rightarrow \lambda^{-T}\tau$

$$\frac{\partial \omega_{\lambda^{-1}a}}{\partial t} + (v_{\lambda^{-1}a} \cdot \nabla) \omega_{\lambda^{-1}a} = (\omega_{\lambda^{-1}a} \cdot \nabla) v_{\lambda^{-1}a} + \nu \lambda^{T-2} \Delta \omega_{\lambda^{-1}a} + M_{\lambda^{-1}a}. \quad (18)$$

At time t , the initial condition (10) defined by $S_{t, \frac{\tau}{\lambda^T}}(\psi_{\lambda^{-1}a}(x, t), F_{\lambda^{-1}a}(x, t))$ is a solution of (18) where the range of the integration over modes for any correlation function is now $[0, \lambda a^{-1}]$. Using (12) and (16) one can substitute $\lambda^{d_\psi} \psi_a(\lambda x, \lambda^T t)$ and $\lambda^{\frac{T}{2}+1} M_a(\lambda x, \lambda^T t)$ for $\psi_{\lambda^{-1}a}(x, t)$ and $M_{\lambda^{-1}a}(x, t)$ in each correlation function. Notice that the fields in each correlation function have now modes only in the range $[0, a^{-1}]$. The integration over the modes in the range $[a^{-1}, \lambda a^{-1}]$ only affects the Boltzmann weight $\exp -S_{t, \lambda^{-T}\tau}(\psi_{\lambda^{-1}a}(x, t), F_{\lambda^{-1}a}(x, t))$. Performing the integration of the Boltzmann weight over these modes gives the renormalised Boltzmann weight

$$\exp -S_{\lambda^T t, \tau}(\psi_a(\lambda x, \lambda^T t), \lambda^{\frac{T}{2}} F_a(\lambda^T t, \lambda x)) = \int_{[\frac{a}{\lambda}, a]} d\psi \exp -S_{t, \frac{\tau}{\lambda^T}}(\psi_{\lambda^{-1}a}(x, t), F_{\lambda^{-1}a}(x, t)) \quad (19)$$

The probability law of the velocity potential (12) is obtained after averaging over the Gaussian stirring force and using (17). The resulting equations for the correlation functions only involves the fields $\psi_a(\lambda x, \lambda^T t)$ and $M_a(\lambda x, \lambda^T t)$. These equations correspond to

$$\frac{\partial \omega_a}{\partial(\lambda^T t)} + \lambda^{2+d_\psi-T} (v_a \cdot \partial_{\lambda x}) \omega_a = \lambda^{2+d_\psi-T} (\omega_a \cdot \partial_{\lambda x}) v_a + \nu \partial_{\lambda x}^2 \omega_a + \lambda^{-1-\frac{T}{2}-d_\psi} M_a \quad (20)$$

when inserted in correlation functions. The derivatives are taken with respect to $\lambda^T t$ and λx . We now require that (20) coincides with (5) at time $\lambda^T t$ and coordinates λx . This is achieved if

$$\begin{aligned} d_\psi &= -1 - \frac{T}{2} \\ d_\psi &= T - 2. \end{aligned} \quad (21)$$

One can then deduce (13). This proves that (12) and (13) specify the time evolution of the probability law of the velocity potential.

Starting from a field theory in the basin of attraction of a fixed point of the renormalisation group, this probability follows renormalisation trajectories. It therefore converges to the probability law specified by the fixed point. The scaling dimension $-\frac{4}{3}$ of the velocity potential becomes its conformal dimension at the fixed point (fixed points of the renormalisation group are conformal theories^[7]). We shall now derive further conditions for the existence of such a fixed point. As the long time regime is linked to the small $\frac{a}{\lambda}$ region by (12), the fixed points satisfy (18) in the limit when λ goes to infinity. Using (13), one can see that the influence of the viscosity becomes negligible in the long time regime as $T - 2 = -\frac{4}{3}$. Similarly, the stirring force can be easily dealt with. Indeed, notice that

$$S_{\frac{L}{\lambda}}(k) = \left(\frac{L}{\lambda}\right)^3 S\left(\frac{kL}{\lambda}\right) \quad (22)$$

converges to zero as $\lambda \rightarrow \infty$. This implies that the 2-point correlation of $F_{\lambda^{-1}a}$ goes to zero and therefore the Gaussian stirring force vanishes in the long time regime. Assuming that the fixed points represent steady states, the time derivatives of each correlation function vanish as well. We therefore find that fixed points are characterised by the vanishing of the non-linear terms in the limit $\lambda \rightarrow \infty$. The non-linear terms can be readily evaluated when the cut-off is rescaled to zero using properties of short distance expansions. Products of fields become singular when the cut-off is removed. For fields called quasi-primary fields^[8], the result of the product can be expanded in a power series in $\frac{a}{\lambda}$. A vanishing limit is obtained if the leading term has a positive exponent. We shall suppose that the velocity potential becomes a quasi-primary field at the fixed point. In that case, the nonlinear terms read

$$\epsilon_{\alpha\beta\gamma}\partial_\gamma((v_{\lambda^{-1}a}\cdot\nabla)v_{\lambda^{-1}a})_\beta \sim \left(\frac{a}{\lambda}\right)^{d_2-4-2d_\psi}\psi_{2,\alpha,\lambda^{-1}a} + \dots \quad (23)$$

where d_2 is the conformal dimension of the leading pseuso-vector $\psi_{2,\alpha}$ in the expansion of the non-linear terms. We can therefore conclude that the fixed points are characterised by the inequality

$$d_2 > 4 + 2d_\psi \quad (24)$$

This generalises a similar inequality obtained by Polyakov in two dimensions. We can now state our main result. Conformal field theories satisfying (24) such that the velocity potential is a quasi-primary field of dimension $-\frac{4}{3}$ describe the long time regime of 3d turbulence.

As a consequence, notice that ψ cannot be identified with ψ_2 as (24) is not satisfied in that case. This implies that

$$\langle \psi_\mu \psi_\nu \psi_\rho \rangle = 0, \quad (25)$$

the three point function of the velocity potential vanishes identically at the fixed point. It is extremely difficult to construct explicitly three dimensional conformal field theories describing an infrared fixed point of the renormalisation group. We shall suppose that such theories exist and deduce consequences on the energy spectrum of the solutions of the Navier-Stokes equation. In particular, we can calculate the 2-point function of the velocity potential and then the energy spectrum in the long time regime. The energy spectrum is easily related to the Fourier transform of the two point function

$$E_a(k) = 4\pi k^2 \int d^3x \langle v_a(x)v_a(0) \rangle \exp -2\pi i k \cdot x. \quad (26)$$

In the long time regime, the energy spectrum behaves as

$$E_a(k) \sim k^{-\frac{5}{3}}. \quad (27)$$

This spectrum is valid for $k \ll \frac{1}{a}$. The exponent is given by $2d_\psi + 1 = -\frac{5}{3}$ ^b. The spectrum coincides with a Kolmogorov spectrum^[9]

^b Our results are compatible with the perturbative calculations of Refs. [2,3,4]. In this case, the stirring force is specified by $\langle F_{a,\alpha}(k, \omega) F_{a,\beta}(k', \omega') \rangle = \frac{W_0}{(k^2+m^2)^{\frac{y}{2}}} \chi_{a^{-1}}(k) (\delta_{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}) \delta(k+k') \delta(\omega+\omega')$

where $\chi_{a^{-1}}(k)$ is equal to one for $k \leq a^{-1}$ and zero otherwise. The mass term $m = L^{-1}$ is small but guarantees that there are no infrared divergences when $k \rightarrow 0$. The same analysis as in the text shows that $d_\psi = -\frac{1+y}{3}$ and the spectrum behaves like $k^{\frac{1-2y}{3}}$ in the range $k \ll a^{-1}$ for $0 < y \leq 5$.

The solutions of the Navier-Stokes equation (12) have some salient features. Let us notice that the evolution of the probability law of the velocity potential (12) makes explicit the breaking of time reversal invariance already present in the Navier-Stokes equation. Indeed, the probability law at time $\lambda^T t$ is obtained from the probability law at time t by averaging over the small scale features of the velocity potential. These small scale features are therefore lost in the process of evolution. In particular, knowing the probability law at time $\lambda^T t$ does not enable one to deduce the probability law at time t . Another consequence is that the renormalisation trajectories converge to a non-unitary conformal field theory as $d_\psi < 0$. This result has also been obtained in 2d^[1]. It reinforces Polyakov's idea that turbulence is a flux state and not an equilibrium state described by an unitary theory. Moreover, at the fixed point the energy spectrum is determined to be a Kolmogorov spectrum. This result highly depends on the fact that the stirring force is Gaussian and acts on large wavelengths. Another feature of our analysis is that we have derived a spectrum compatible with a Kolmogorov spectrum without any cascade hypothesis^[9]. Finally, let us note that our results can be applied in two dimensions, in particular (24) is replaced by Polyakov's condition^{[1]c} and the energy spectrum is still given by a Kolmogorov spectrum.

We have been able to obtain properties of solutions of the 3d Navier-Stokes equation in the long time regime. Assuming that the probability law of the stream function is a statistical field theory, its time evolution corresponds to renormalisation trajectories. Starting from initial conditions in the basin of attraction of a fixed point under renormalisation transformations, the solutions converge asymptotically to the fixed point. The possible fixed points are all non-unitary. In the case where the random stirring has a spectrum concentrated on large wavelengths, we have shown that a Kolmogorov spectrum is obtained in the long time regime. The existence of such a fixed point of the renormalisation group governing the long time regime of turbulence seems physically clear in view of the numerous experimental as well as computational evidence in favour of a Kolmogorov spectrum. It is a challenging problem to construct such a field theory.

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^c The stream function ψ satisfies $\psi_{\lambda^{-1}a}(x)\psi_{\lambda^{-1}a}(x) \sim (\frac{a}{\lambda})^{d_2-2d_\psi}\psi_{2,\lambda^{-1}a}(x)$ with $d_2 > 2d_\psi$.

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